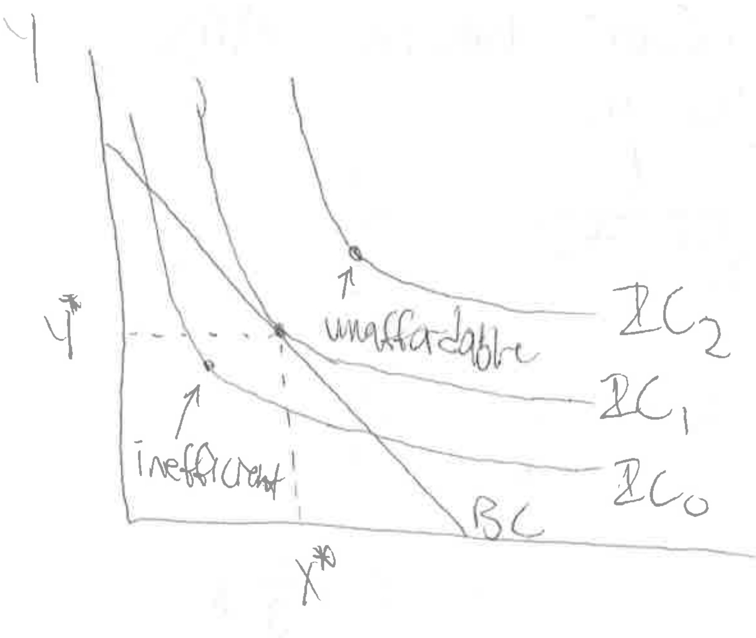
Putting everything together

Lecture 4 Jan 27

Solving the consumer's optimization problem.

Consumer's choose the "best" bundles of goods that they "can afford."

$$\left\{ \begin{array}{ll} \max\limits_{\{X,Y\}} & U(X,Y) \\ \text{s.t.} & M = P_X X + P_Y Y \end{array} \right\}$$




Tangency is the key to finding the optimal bundle.

- where the slope of the IC is equal to the slope of the budget constraint
- i.e. where $\boxed{MRS_{XY} = \frac{P_X}{P_Y}}$

$$\frac{MU_X}{MU_Y} = \frac{P_X}{P_Y}$$

$$\Rightarrow \frac{MU_X}{P_X} = \frac{MU_Y}{P_Y}$$

# Constrained Optimization Example

Sarah gets utility from ~~soda~~ coke $(x)$ and hotdogs $(y)$. Her utility is given by $U = x^{1/2} y^{1/2}$. Her income is \$12, and the prices of coke and hotdogs are \$2 and \$3, respectively. What is Sarah's utility-maximizing bundle of coke and hotdogs

$$\max_{\{x,y\}} x^{1/2} y^{1/2} \quad \text{st } 12 = 2x + 3y$$

## Method 1: Tangency Condition

The point of tangency occurs where $MRS_{xy} = \frac{P_x}{P_y}$

$$MRS_{xy} = \frac{MU_x}{MU_y} = \frac{\cancel{\frac{1}{2}} x^{-1/2} y^{1/2}}{\cancel{\frac{1}{2}} x^{1/2} y^{-1/2}} = \frac{y}{x}$$

Tangency condition $\Rightarrow \frac{y}{x} = \frac{2}{3}$

$\hookrightarrow y = \frac{2}{3} x$

Plug back into B.C.

$$12 = 2x + 3y$$
$$= 2x + \cancel{3}\left(\frac{2}{\cancel{3}} x\right)$$

$4x = 12$

$x^* = 3 \qquad y^* = 2 \qquad \boxed{(x^*, y^*) = (3, 2)}$

$y = \frac{2}{3}(3)$

$$u(x^*, y^*) = (3)^{1/2} (2)^{1/2} \approx 2.449$$

## Method 2: Method of Lagrange

$$\max_{\{x,y\}} x^{1/2}y^{1/2}$$
$$12 = 2x + 3y$$



step 1) set up Lagrangian function

step 2) Take first-order conditions w.r.t. choice variables and Lagrange multiplier

step 3) solve a system of equations

Step 1

$$\mathcal{L} = x^{1/2}y^{1/2} + \lambda(12 - 2x - 3y)$$

Step 2

FOC's

$[x]$ $\frac{1}{2}x^{-1/2}y^{1/2} - 2\lambda = 0$ 1

$[y]$ $\frac{1}{2}x^{1/2}y^{-1/2} - 3\lambda = 0$ 2

$[\lambda]$ $12 - 2x - 3y = 0$ 3

3 eqns
3 unknowns

Step 3

- divide 1 by 2 $\Rightarrow$ $\frac{\cancel{\frac{1}{2}}x^{-1/2}y^{1/2}}{\cancel{\frac{1}{2}}x^{1/2}y^{-1/2}} = \frac{2\cancel{\lambda}}{3\cancel{\lambda}}$

$\Rightarrow \frac{y}{x} = \frac{2}{3}$

$\hookrightarrow y = \frac{2}{3}x$

- plug into 3:

$12 - 2x - \cancel{3}\left(\frac{2}{\cancel{3}}x\right) = 0$

$4x = 12 \Rightarrow x^* = 3 \quad y^* = 2$

Method 3: substitution

- plug the constraint into the objective function

$$\begin{cases} \max\limits_{\{x,y\}} & x^{1/2}y^{1/2} \\ & \text{s.t. } 12 = 2x + 3y \end{cases}$$

$$12 - 2x = 3y$$
$$y = 4 - \frac{2}{3}x$$

$$\begin{cases} \max\limits_{\{x\}} & x^{1/2}\left(4 - \frac{2}{3}x\right)^{1/2} \end{cases}$$ - unconstrained optimization

$$\frac{\partial}{\partial x} = x^{1/2} \cdot \frac{1}{2}\left(4 - \frac{2}{3}x\right)^{-1/2}\left(-\frac{2}{3}\right) + \left(4 - \frac{2}{3}x\right)^{1/2} \cdot \frac{1}{2}x^{-1/2} = 0$$

$$\frac{1}{3}x^{1/2}\left(4 - \frac{2}{3}x\right)^{-1/2} = \frac{1}{2}x^{-1/2}\left(4 - \frac{2}{3}x\right)^{1/2}$$

$$\frac{2}{3}x = 4 - \frac{2}{3}x$$

$$\frac{4}{3}x = 4 \Rightarrow x^* = 3$$

$$y = 4 - \frac{2}{3}(3)$$

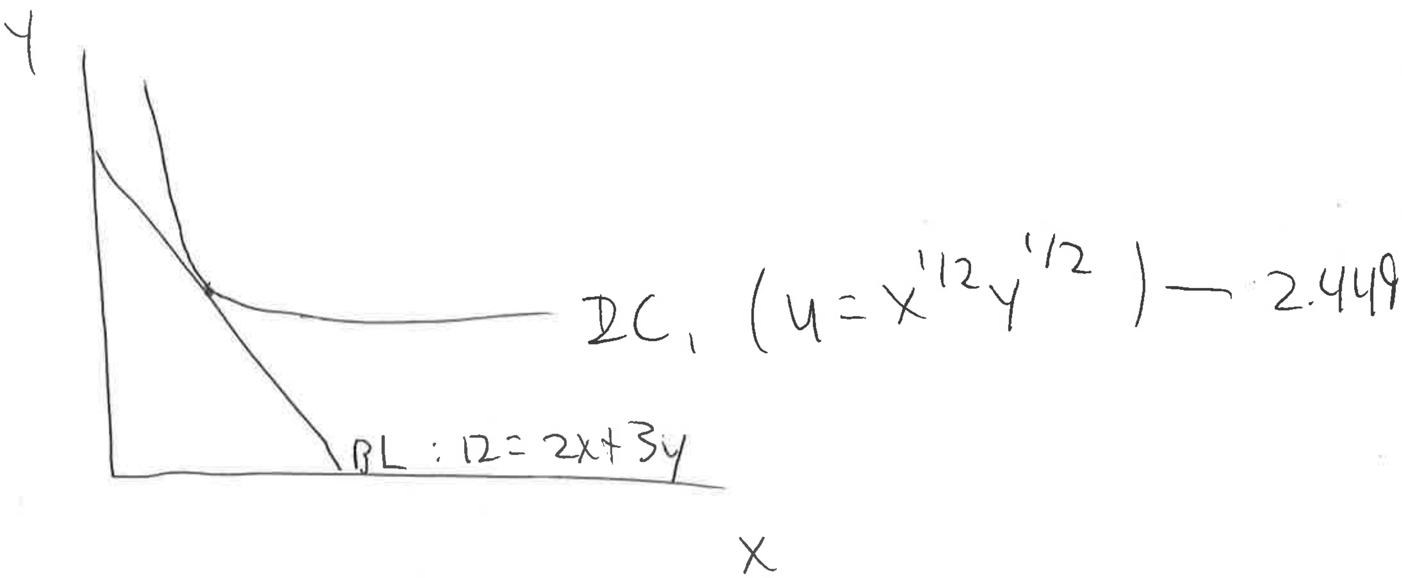
$$y^* = 2$$

$$\begin{cases} \max\limits_{\{x,y\}} u(x,y) \\ \text{s.t. } m = P_x x + P_y y \end{cases}$$

↳ "dual problem" Expenditure minimization

↳ $\min \; P_x x + P_y y$

s.t. $\cancel{x^{1/2}y^{1/2}} = u(x,y) = \bar{u}$




$$\min_{\{x,y\}} E = 2x + 3y$$

s.t. $x^{1/2}y^{1/2} = 2.449$

$$MRS_{xy} = \frac{P_x}{P_y} \Rightarrow \frac{\frac{1}{2}x^{-1/2}y^{1/2}}{\frac{1}{2}x^{1/2}y^{-1/2}} = \frac{2}{3}$$

$$y = \frac{2}{3}x$$

$$x^{1/2}\left(\frac{2}{3}x\right)^{1/2} = 2.449$$

$$\left(\frac{2}{3}\right)^{1/2} x = 2.449$$

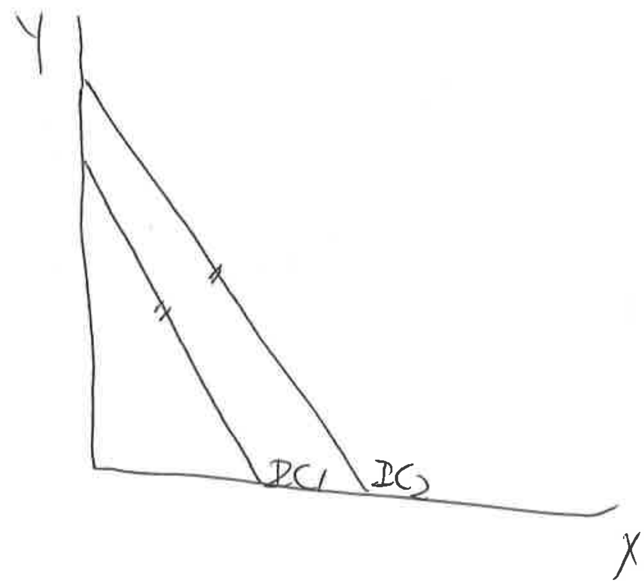
$$\boxed{\begin{array}{l} x^* = 3 \\ y^* = 2 \end{array}}$$

Sometimes standard optimization methods will not work.
This can occur when 1) you don't get an interior solution
2) your budget constraint or indifference curve is not diff'ble.

## Example of Perfect substitutes

Let $U = x + y$, $P_x = 2$, $P_y = 2$, $M = 12$

Step 1 Draw indifference curves




$u = x + y$

$y = u - x$

Step 2 Draw the Budget constraint

$12 = 2x + 2y$

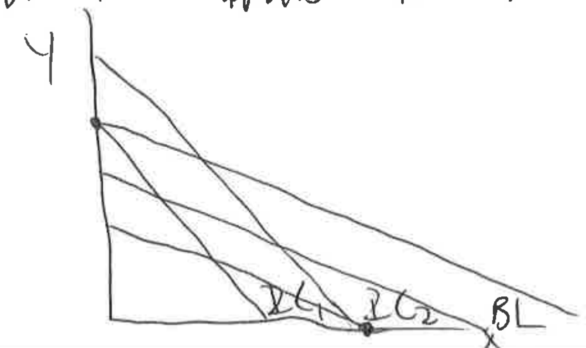
$12 - 2x = 2y$

$y = 6 - x$

In this case, the Budget line overlaps w/ the indifference curve.

slope of ICs $= -1 =$ slope of BC.

What happens if $P_x = 1$, $P_y = 2$, $M = 12$



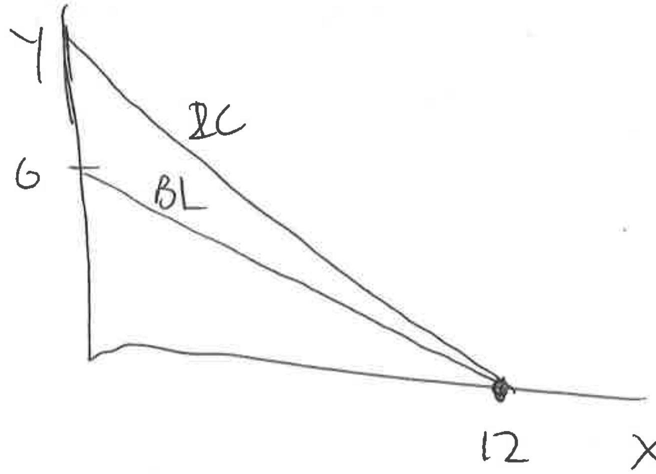

$12 = x + 2y$

$12 - x = 2y$

$y = 6 - \frac{1}{2}x$

BC slope $= -\frac{1}{2}$ (flatter)

Rule: when the IC is everywhere steeper or flatter than the BC you will have a corner solution.

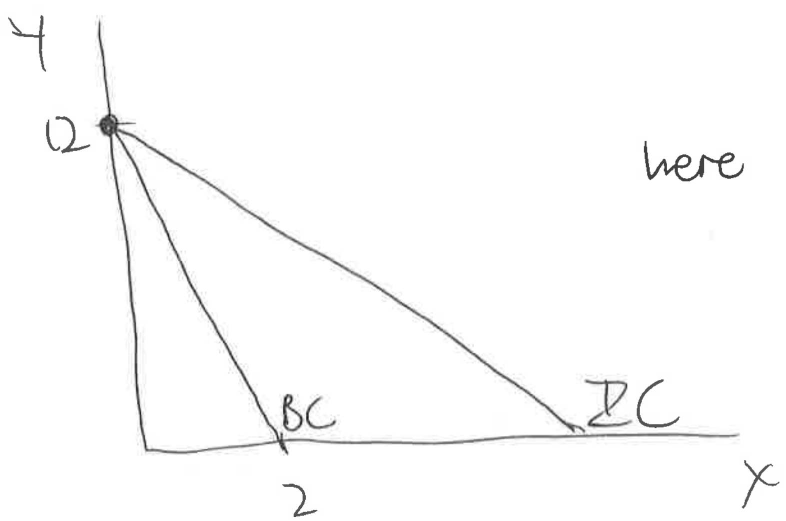



Here the optimal bundle is $(x^*, y^*) = (12, 0)$

Ex: $U = 4x + 2y$ $\quad P_x = 6$, $P_y = 1$ $\quad M = 12$

$u - 4x = 2y$ $\qquad 12 = 6x + y$

$y = \frac{u}{2} - 2x$ $\qquad y = 12 - 6x$

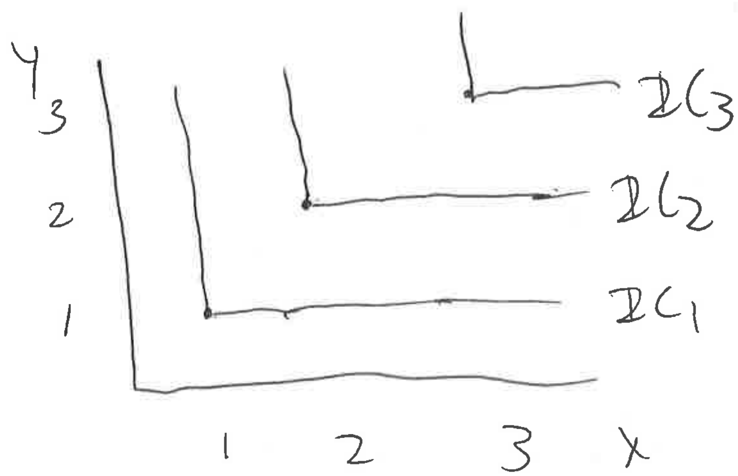



here optimal bundle is $(0, 12)$

# Example of Perfect Complements

shampoo (x) [mitzi] and conditioner (y) perfect complements at a 1:1 ratio. $u = \min\{x, y\}$

a) draw indifference curves



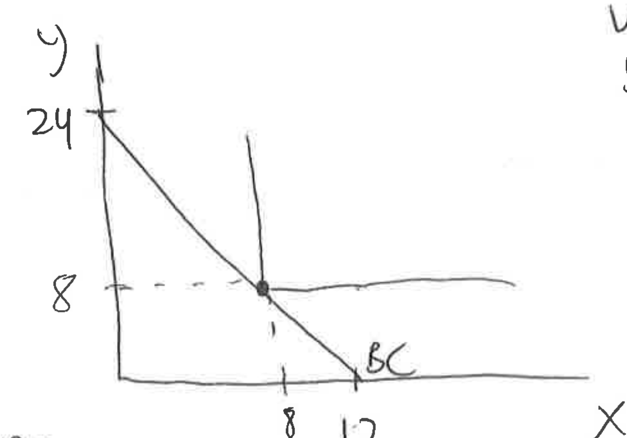

b) shampoo costs \$4 & conditioner \$2, draw BC. what is optimal bundle?

spps $m = 48$

$48 = 4x + 2y$

$48 - 4x = 2y$

$y = 24 - 2x$




we have to consume shampoo & conditioner in the same proportion

$u = \min\{x, y\}$

$x = y$

$x = 24 - 2x$

$3x = 24 \Rightarrow x^* = 8$

$y^* = 8$

c) spps prices change, shampoo costs \$2 and conditioner costs \$4, what is likely to happen to the optimal bundle. Nothing should happen since we must consume shampoo & conditioner in equal proportions. (5)

$48 = 2x + 4y$

$4y = 48 - 2x$

$y = 12 - \frac{1}{2}x$

$u = \min\{x, y\}$

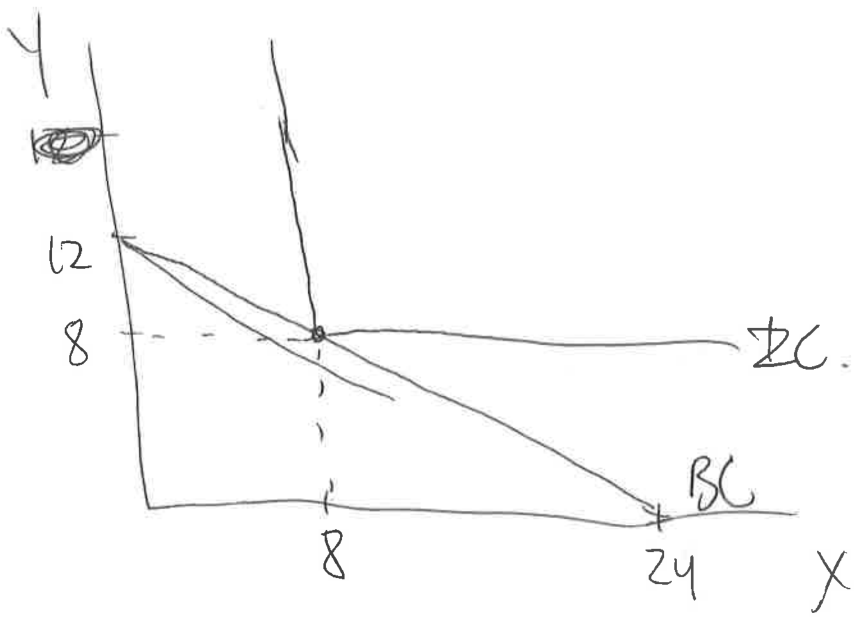
$x = y$

$x = 12 - \frac{1}{2}x$

$\frac{3}{2}x = 12$

$x^* = 8$

$y^* = 8$



d) How would the optimal bundle from (c) change if Mitzi uses 2 squirts of shampoo & 1 of conditioner?

$$48 = 2x + 4y$$

$$y = 12 - \frac{1}{2}x$$

$$\frac{x}{y} = \frac{2}{1}$$

$$\Rightarrow x = 2y$$

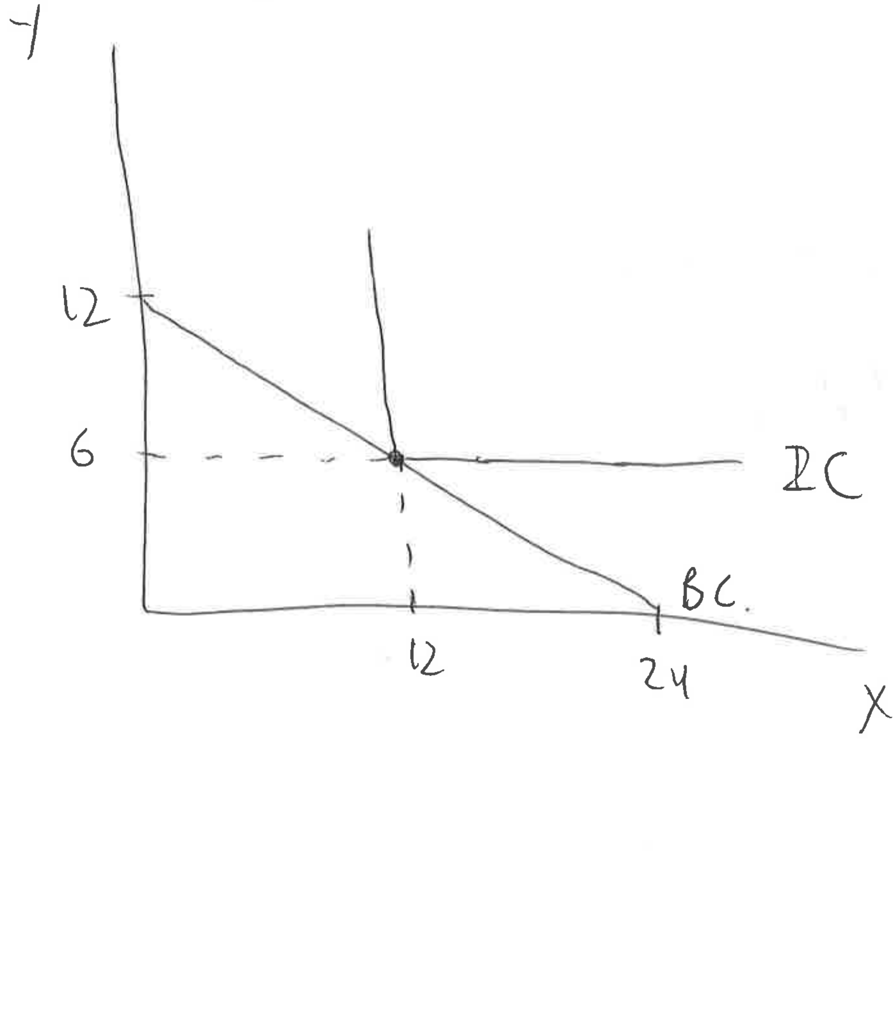
$$y = 12 - \frac{1}{\cancel{2}}(\cancel{2}y)$$

$$2y = 12$$

$$y^* = 6$$

$$x = 2(6)$$

$$x^* = 12$$

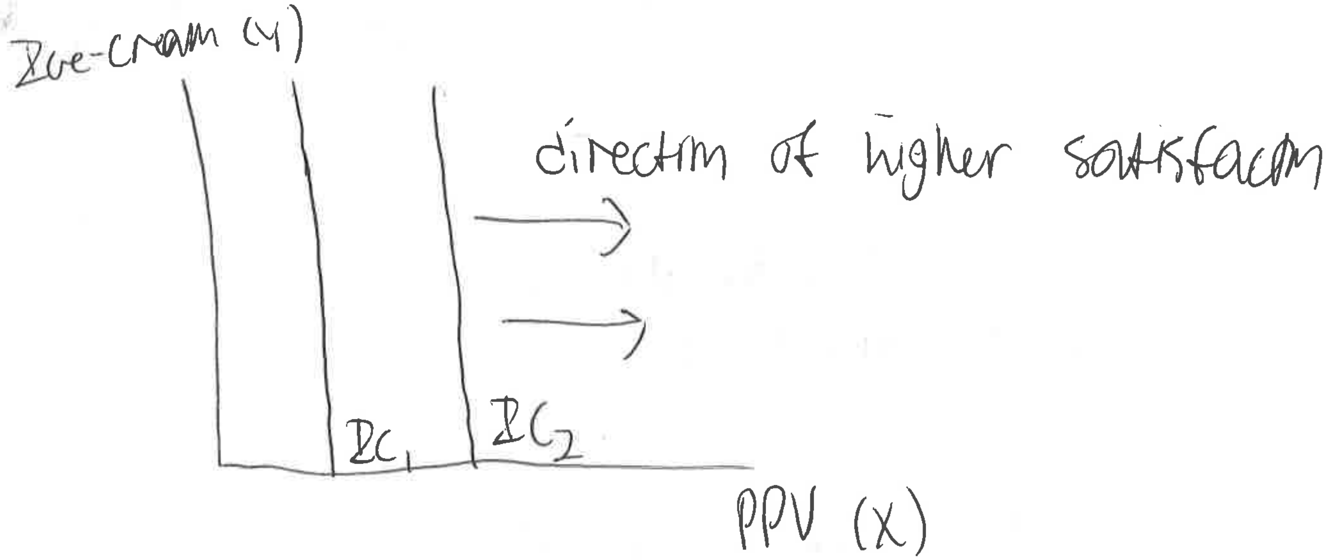


Rule: When the IC is everywhere steeper or flatter than the BC you will have a corner solution.

Ex

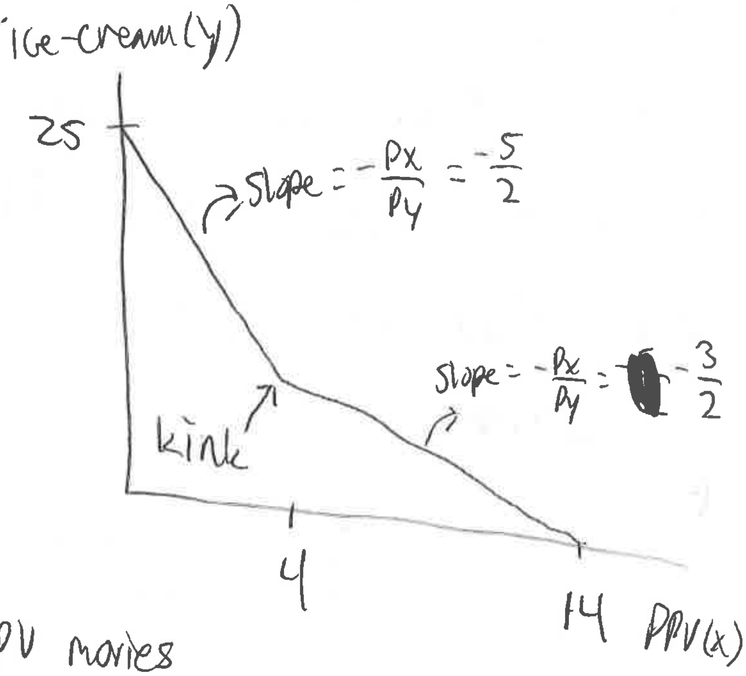
Consider a world with pay-per-view movies (x) and ice cream (y).

a). Alex likes PPV movies but is indifferent towards ice-cream. Draw his Indifference curves and show the direction of higher satisfaction



b) Ice Cream costs \$2 per quart and Comcast charges \$5 per order of PPV movies for the first 4 orders and \$3 per order for additional orders over 4. If Alex has \$50 to spend, graph his B.C.

$$P_x = \begin{cases} 5 & \text{first 4} \\ 3 & \text{every order after} \end{cases}$$

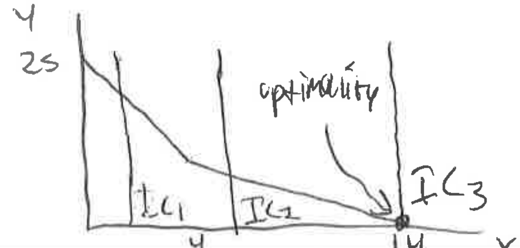



$$50 = 5 \cdot 4 + 3 \cdot z$$
$$30 = 3z$$
$$z = 10$$

So the most PPV movies he can buy is $10 + 4 = 14$

c) Find Alex's best feasible bundle.

From our rule, if ~~every~~ the IC is everywhere steeper than the BL, we are at a corner solution. From (a), our IC has a slope of $\infty$, thus it is everywhere steeper. Alex will consume at the corner that places him at the highest IC, i.e. (14, 0)




Here, $IC_3$ is the highest IC.